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Sustainability: Dynamics and Uncertainty

Graciela Chichilnisy,Geoffrey Heal, and Allessandro
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Editors

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RALPH ABRAHAM, GRACIELA CHICHILNISKY AND RON RECOR

2.2. North-South Trade and the Dynamics of the Environment

1. Introduction

This paper develops a dynamic model of North-South trade in w
environment plays an important role. Our model is based on Chich
North-South model for the macroeconomic interaction between twc
of the world economy. The latter was introduced in a static conte›
We introduce dynamics in the original North-South model by allo
endogenous accumulation of capital. As a second extension of [1],
duce here a variable which represents the system of property righ
environmental asset which is used as an input to production.[1] This
could represent, for example, the property rights on forests from whi
is extracted to be used as an input to the production of traded good
property rights on water which is similarly used, perhaps for agr
goods for export.

The paper explains mathematically and through simulations the d
of a two-region world. There are two produced goods and two i
production. Capital is one input: it accumulates in the two regions
time as a function of profits. We show that as we vary the property
the environment the dynamics of the system changes. The less well
are the property rights, the more chaotic are the model's dynamics.

The models which result bear some similarity to one created by J
Neumann in 1932 and extended by Richard Goodwin in 1990 [12, ch
We establish, in a sequence of steps, that these models are varian
coupled logistic maps studied in several recent papers, for example, [
idea is to alter [1] to allow capital accumulation through time, assun
the approach to equilibrium follows rapidly. New equations are intro
our model, which are not found in [1] or [3]. These equations des
evolution of capital stock through time by accumulation and depreci

G. Chichilnisky et al. (eds.), Sustainability: Dynamics and Uncertainty, 77–107.

The outline of the paper[3] and the main results are as follows. In Section 2 we introduce some useful notation, and in Section 3 the static North-South model [1] is recalled. Following that, we develop in Section 4 a rather simple one-dimensional model which is pedagogically useful because it anticipates the mathematical structure of our main model. We analyze its dynamical behavior in a sequence of propositions, and confirm this behavior through simulation. This dynamical behavior is essentially equivalent to the logistic map, and is similar to that which will be found later in our main model. In Section 5.2 we introduce our main (two-dimensional) model, and establish its dynamic behavior through simulation. We find a very rich dynamic behavior, with an extensive web of bifurcations controlled by the environmental property rights parameters. We find chaotic attractors, and chaotic separatrices. That is, the basins of attraction form a fractal structure. In Section 6, the conclusions, we interpret our results in the broader context of North-South trade and the environment.

1.1. *The Dynamic North-South Model*

Our dynamic model is based on [1], but with a major extension. Two fundamental equations are added to those of [1], which endogenize the changes in capital stock in the two regions through time. We first explain intuitively how the dynamical model is defined, and following this we provide the mathematical definitions.

The dynamical model is constructed iteratively as follows. Start from given values of the exogenous parameters of the North-South model[4] of [1]. The vector of initial levels of capital stocks in the two regions is a two-dimensional parameter, which will be the initial value (for $t = 1$) of our dynamical system in the plane. Now solve the static North-South model analytically.[5] The solution gives us, *inter alia*, the equilibrium value of *GNP* in each region.[6] So far the model is static, and identical to that in [1]. How does our dynamical system move in the plane from period $t = 1$ to period $t = 2$? To define the dynamics we will introduce two new equations, one in each region, both depending on the corresponding equilibrium level of *GDP* in the region in period $t = 1$. These equations explain how capital accumulates: a proportion of *GDP* in $t = 1$ is saved and increases previous period capital stock, while some of the old capital depreciates. From these equations one updates capital stocks and obtains a new set of exogenous parameters for the (static) North-South model for $t = 2$. These differ from the previous set (for $t = 1$) only with respect to the initial capital stocks, which have now varied according to our two new equations. The new capital stocks for the North and the South define a two-dimensional vector describing the period $t = 2$ value of our dynamical system. Now solve the (static) North-South model for this new set of exogenous parameters, and obtain *GDP* for period $t = 2$. Iterating this procedure defines the dynamical system in the plane for every period $t \geq 1$.

The following is the math
above.

Our first goal is to define
add to the equations of the
two new equations, a two-
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$K_N(t+1)^+ = s_N$

$K_S(t+1)^+ = s_S$

Equation (1.1.1) describ
and (1.1.2) in the South.
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(N) as the sum of capital
minus the part of this whi
the North) plus *savings*, w
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Equations (1.1.1) and (1.1.
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(1.1.1) and (1.1.2).

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are the *international terms*
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The following is the mathematical formulation of the procedure explained above.

Our first goal is to define the two new capital accumulation equations which add to the equations of the (static) North-South model and obtain, from these two new equations, a two-dimensional discrete dynamical system, generated by an endomorphism of the plane, $T : \Re^2 \mapsto \Re^2$. The two new equations are:

$$K_N(t+1)^+ = s_N(GNP_N) + (1-\delta_N)K_N(t), \tag{1.1.1}$$

$$K_S(t+1)^+ = s_S(GNP_S) + (1-\delta_S)K_S(t). \tag{1.1.2}$$

Equation (1.1.1) describes capital accumulation through time in the North, and (1.1.2) in the South. These equations are standard, and are interpreted as follows. Equation (1.1.1) explains *capital stock* at time $t+1$ in the North (N) as the sum of capital stock in the previous period in the North, $K_N(t)$, minus the part of this which is depreciated (δ_N is the depreciation factor in the North) plus *savings*, which is the savings rate in the North, s_N, times the *gross national product* in the North, GNP_N.

In order to determine our two-dimensional discrete dynamic system we need to define from these equations an endomorphism of the plane, $T : \Re^2 \mapsto \Re^2$. The depreciation and savings rate are exogenously given parameters. But how do we determine *GNP* in the two regions for any given values of the capital stocks in each, considering that they trade with each other through the international market?

The solution to this problem is one of the main contributions of our paper: the specifications of the *GNP* variables as the solutions of two simultaneous market equilibrium problems. Here is where we use [1]. The combination of Equations (1.1.1) and (1.1.2) with the North-South trade model is done here for the first time, and we call this the *dynamic North-South model.*

How do we obtain an endomorphism of the plane from the two equations for capital accumulation? We start with initial values of the two capital stocks, one for each region, K_N and K_S. The *static* North-South model solves the world economy equations from the following initial parameters: *capital* and *labor supply*, *technologies* and *demand* in each region. Here, for the dynamic North-South model, we assume instead that *capital* and *labor supply* and *technologies* are initially given in each region.

In each region, at time t, we solve fully the static North-South model at time t and obtain *GNP* at time t. From this, in turn, we compute the capital stocks, at time $t+1$, using our new dynamic equations for capital accumulation, (1.1.1) and (1.1.2).

The procedure can be summarized as follows. The static North-South model determines endogenously five price variables and sixteen quantity variables. It has two goods traded internationally (basic goods, B, and industrial goods, I) and two factor of production (capital, K, and labor, L). The price variables are the *international terms of trade* for the two traded goods B and I, denoted by p_B and p_I, (these are reduced to one by the normalizing assumption $p_I = 1$, and henceforth $p = p_B$), and the *prices of labor* and *rental of capital* in each region, denoted w and r. Technologies are different in the two regions so that the rewards to labor and to capital are also different. The sixteen quantities which are endogenously determined are: *supply* and *demand* for the basic and industrial goods, *employment of factors* in the two sectors, imports and exports of both goods, all in each of the two regions. From these endogenous variables we obtain an expression for the desired *GNP* in each region. By definition, *GNP* is the value of the gross national product, that is, the value of all outputs minus all inputs (of B and I) computed at the equilibrium market prices, p. These are the prices at which all markets clear. Recall that part of the production of each country is consumed in the other country, and that relative prices p have adjusted to permit this trade and to clear markets, so that imports equal exports in each of the two traded goods. The result is an equilibrium level of *GNP* in each region,

$$GNP_N = pB_N^S + I_N^S, \tag{1.1.3}$$

$$GNP_S = pB_S^S + I_S^S. \tag{1.1.4}$$

Here p, B^S, and I^S are determined as the solution of a system of 22 simultaneous equations in 22 variables, as in the *static* North-South model, This is explained in Section 5.2 below. *Therefore, for each value of capital stock we have assumed an instantaneous adjustment to an equilibrium in the static North-South model.*

From all this we obtain the *GNP* in each region at time t. The two dynamic equations (1.1.1) and (1.1.2) then provide capital stocks in the two regions at the next period, $t+1$. Our plane endomorphism, T, is now well defined.

The equations describing *GNP* in each region are nonlinear. Therefore, the endomorphism T is nonlinear as well. In the following we shall study its qualitative properties and experiment with simulations depicted graphically. But before analyzing the model, it will be useful to explain the connections with the environment.

1.2. *North-South Trade and the Environment*

The environment appears in this model as one of the inputs, or *factors* of production. While in the original North-South model the two factor of production are *labor* and *capital*, recently [4] the model has been extended to three factors of production, one of which is a *natural resource*, such as water from an aquifer, or fish from a common body of water, or wood from a common forest. In the original North-South model the behavior of a certain parameter α – representing the supply response of a factor to its price – is shown to be crucial in explaining the patterns of trade between the two regions, including the terms of trade and the gains from trade. Furthermore, in [4], the absolute value of this parameter in the South, α_S, is proven to vary with the *property* *rights regime* for the resource (such as land). This resource is used as an input for the production of the traded goods (such as cash crops: coffee, cotton palm oil). It is, therefore, of interest to simulate the behavior of the North-South model with *different property rights* for this environmental resource, that is, different values of α_N and α_S. These parameters contain crucial information about property rights. It was shown in [4–6] that α_S is smaller when the property rights are well defined, and is larger when they are ill-defined. As an example, [4] predicts that a regime of property rights which gives better rights to the locals of the rainforest (for example, in Guatemala and Ecuador) could improve the terms of trade on cash crops and control the overexploitation of its rainforest.

We now apply our model to explain the fundamental connection between the environment and trade. We will look at the environment as a common property resource which is used as an input to production in both regions. Examples are: rainforests, bodies of water, or fisheries. These are inputs to the production of environmentally intensive goods which are internationally traded, such as: wood products, industrial output, cash crops (cotton, coffee, soya beans, palm oil). In our model, we shall now reinterpret L as an *environmental input* used, together with the other input, K, to produce basic and industrial goods, B and I. Thus, we rename L as E for the remainder of this section.

As already mentioned, a crucial parameter in the North-South model is α, the response of the supply of E to it relative price, w/p. In [1] and [3], this parameter was shown to determine the properties of the solutions (equilibria). Here, α will play a similar role: it represents the *property rights* on the environmental resource, E: α is smaller when the property rights are well defined, and larger when they are ill-defined. For example: if the local population has well-defined property rights on the biodiversity of a rainforest, which is an input to the production of pharmaceuticals, then the wood input E will be harvested more carefully. Obtaining a larger supply of E requires a larger increase of the price of E, p_E. Thus, α is smaller when the property rights on the rainforest are well defined. The theory and the analytics proving this fact appear in [5] as lemma 1. When property rights on the rainforest are ill-defined, α is large: this means that a lot more wood will be harvested, and the forest may be destroyed, for smaller increases in prices. The *price* represents the value of the input.

It has been shown in [4, 6] that well-defined property rights lead to better valuation of scarce resources. Good examples are provided by Merck Pharmaceuticals, Inc. and Shaman Pharmaceuticals, Inc. These companies have entered into agreements to advance cash and to share the profits from prospecting biodiversity samples in Costa Rica and in South American countries. The biodiversity samples are an input to the production of valuable pharmaceuticals (examples: *curare* and the more recently discovered *periwinkle* which treats Hodgkins disease and leukemia in children) sharing the profits with the locals. This amounts to improving the property rights of the local population on the common property resource: the rainforest's biodiversity. This scheme is not too different from the venture capital agreements which advance working capital to use intellectual property (software ideas) and share the rights subsequently with the entrepreneurs. By increasing the realized value of the common property input, these agreements increase the interest in conservation by those who would otherwise overuse or overexploit the resource beyond its biological steady-state extraction rate.

All of these considerations may be represented in the North-South model by varying the parameter α_S in the South. This variation simulates the input of property right agreements in developing countries for their valuable common property resources. For the theories explaining the general impact of varying in the static North-South model in [3], see [5]. In this paper we address the *dynamic* North-South model, and ask the same questions. The problem is more complex since our model is dynamic, and we rely on simulation to provide our answers.

1.3. *Organization of the Paper*

We begin by recalling the static North-South model. Then we will develop the equations for the general form of the dynamic North-South model in the sequence of steps. To reveal the mathematical structure of the problem, we will present, in the first of these steps, a very simplified one-dimensional dynamic version of our two-dimensional dynamic system. This is only a mathematical artifice, as the economics are embodied only in the full two-dimensional version, our main model, of Section 5.2. We then explain some properties of the dynamical model and present simulations which confirm our results and suggest possible extensions. We end with a proposal for a dynamical system linking our dynamic North-South model with the atmospheric chemistry of the carbon cycle.

2. Notational Conventions

We will write K_N in place of $K(N)$ used in [3]. We are going to encounter symbolic expressions in the variables:

$$K_N, K_S, s_N, s_S, \ldots,$$

and so on. We will refer to K for example as a *root symbol*, and only when accompanied by a subscript N or S will the symbol denote a variable. Thus, we may write expressions or equations in these root variables, but they are symbolic only. When the appropriate subscripts are adjoined, they become expressions or equations of variables defined in our models. Let A be an expression of root symbols. Then A_N will denote the same expression in

the corresponding variable
South, while A_T will be de
Note: Equation (GC2.21

3. Recalling the North-S

We begin with the parame
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$$B^S = \min(L/a_1,$$

the corresponding variables of the North system, and likewise for A_S for the South, while A_T will be defined to mean $A_N + A_S$.

Note: Equation (GC2.21b) denotes equation 2.21b in [3].

3. Recalling the North-South Model

We begin with the parameters, variables, and notations of the static North-South model as defined in [3]. The root symbols of the eight parameters in each region are: $a_1, a_2, c_1, c_2, \alpha, \beta, \bar{K}$ and $\bar{L}$. Thus, we will encounter $a_1 = a_{1N}, a_{1S}$, etc. The crucial variables which determine the model are five price variables and sixteen quantity variables. The price variables are:

1. $p = p_B$ denotes the *price* of *basic goods*, B. Since the *price* of *industrial goods*, I, had been set to unity, $p_I = 1$, p is the *relative price* of *basics with respect to industrial goods*. It is also called the *terms of trade* since B and I are the only two goods in the international market. In a market equilibrium, p is the same in both regions, North and South, but all other price variables may differ in the two regions.
2. w denotes *wages*.
3. r denotes the *capital rental price*.
   Since labor and capital are *not* traded internationally (that is, between the two regions), their values are determined by p according to local conditions (Equations GC2.21b, GC2.4a) which are unequal in the two regions (because two regions have different production technologies). The five price variables, or *prices*, are p, r_N, r_S, w_N, w_S.
   The *quantity variables* are the following.
4. K denotes *capital stock*. This is determined by r, see (GC2.4) and Figure 1. This relationship is for the static model only. This K will be determined, in the dynamic models of this paper, by a discrete dynamical system modeling the annual variation of capital stock in each region.
5. L denotes *labor*. This is determined by w and p, see (GC2.3).
6. B^S and B^D denote quantities of *basic goods supplies* and *basic goods demanded.*
7. I^S and I^D denote quantities of industrial goods supplied and demanded.
8. $X_B^S = B^S - B^D$ and $X_I^S = I^S - I^D$ denote exports of goods, the excess of what is supplied over what is consumed in each region.

The sixteen quantity variables are: $L, K, B^S, B^S, I^S, I^D, X_B^S, X_I^S$, in each region. The diagram of Figure 1 shows how p (and the parameters in each region) determine all of these other variables. Labor, L, and capital, K, are the inputs to production. Using labor can capital the two economies produce the two goods, or commodities, B^S and I^S. In each region, B^S is produced using labor and capital according to

$$B^S = \min(L/a_1, K/c_1). \tag{3.1}$$

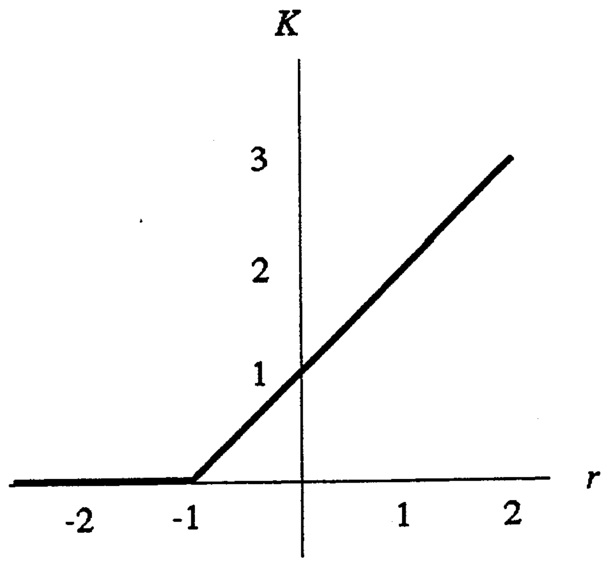

Figure 1. Grahp of $K(r)$. The y-intercept is at $\bar{K}$, and the slope is β.

Therefore, efficient use of L and K requires that

$$B^S = L/a_1 = K/c_1,$$

that is, labor and capital are used in fixed proportions for each level of output of B^S, or

$$L/K = a_1/c_1,$$

where a_1 is called the *labor-output ratio* (since $B^S = L/a_1$) and c_1 is called the *capital output ratio* (since $B^S = K/c_1$). Equation (3.1) is the *production technology* which determines how much B can be produced with the available K and L. Similarly, each region has a production technology for I,

$$I^S = \min(L/a_2, K/c_2) \tag{3.2}$$

with the same interpretation for the parameters a_2 and c_2. Equations (3.1) and (3.2) give rise to (GC2.20). (GC2.20 indicates equation number 20 from section 2 of [3].)

Now α and β represent the responses of labor and capital supplies to changes in their prices: w and r. We postulate:

$$L = \alpha w/p_B + \bar{L} \tag{GC2.3}$$

with $\bar{L} < 0$, and

$$r = (K - \bar{K})/\beta \tag{GC2.4}$$

with $\bar{K} > 0$. Equation (GC2.:
does the supply of labor. An
The negative value of $\bar{L}$ inc
before people supply positiv
Note: these relationships
paper, while retaining the stat
with a dynamic rule of capit
Some further relationships

$p_B = (a_1 - rD)/a_2$

$B^S = (c_2L - a_2K)$,

$I^S = (a_1K - c_1L)/$

$w = (p_Bc_2 - c_1)/D$

all of which are non-negative

$D = a_1c_2 - a_2c_1.$

All remaining symbols deno
script S in B^S and I^S and de
Also the subscript B in p_B in
Henceforth, we will omit the
Section 4). Hence: L for L^S
p_I), B for B^S (we will not us
mean demand for industrial g

$p = (a_1 - rD)/a_2$

$B = (c_2L - a_2K)/1$

$I = (a_1K - c_1L)/L$

$L = \alpha w/p + \bar{L}$

$w = (pc_2 - c_1)/D$

$r = (K - \bar{K})/\beta$

all non-negative, and

$D = a_1c_2 - a_2c_1.$

To close the static model in [
$I = \bar{I}^D$
exogenously in each region.
This "closure" correspon
assuming a simple preferen

with $\bar{K} > 0$. Equation (GC2.3) means that as the real wage w/p_B increases, so does the supply of labor. And Equation (GC2.4) means the same for capital. The negative value of $\bar{L}$ indicates the minimum wage needed for survival before people supply positive labor.

Note: these relationships are particular to the static model. Later in this paper, while retaining the static relationship (GC2.3), we shall replace (GC2.4) with a dynamic rule of capital accumlation.

Some further relationships are the following, all from [3]:

$$p_B = (a_1 - rD)/a_2, \tag{GC2.21}$$

$$B^S = (c_2 L - a_2 K)/D \tag{GC2.20}$$

$$I^S = (a_1 K - c_1 L)/D \tag{GC2.20}$$

$$w = (p_B c_2 - c_1)/D, \tag{GC2.21}$$

all of which are non-negative, and

$$D = a_1 c_2 - a_2 c_1.$$

All remaining symbols denote constants defined in [3]. Note that the superscript S in B^S and I^S and denotes Supply (vs Demand), not South (vs North). Also the subscript B in p_B indicates Basic (vs the subscript I for Industrial). Henceforth, we will omit these subscripts when no confusion results (esp. in Section 4). Hence: L for L^S (we will not use L^D), p for p_B (we will not use p_I), B for B^S (we will not use B^D), and I for I^S (we will write I^D when we mean demand for industrial goods). Thus the equations above become:

$$p = (a_1 - rD)/a_2 \tag{GC2.21a}$$

$$B = (c_2 L - a_2 K)/D \tag{GC2.20a}$$

$$I = (a_1 K - c_1 L)/D \tag{GC2.20b}$$

$$L = \alpha w/p + \bar{L} \tag{GC2.3a}$$

$$w = (p c_2 - c_1)/D \tag{GC2.21b}$$

$$r = (K - \bar{K})/\beta \tag{GC2.4a}$$

all non-negative, and

$$D = a_1 c_2 - a_2 c_1.$$

To close the static model in [3], two more variables were fixed:

$$I = \bar{I}^D$$

exogenously in each region.

This "closure" corresponds to the demand specification derived from assuming a simple preference form, which was defined and illustrated in

[3]. One can consider several other demand specifications without changing the structure of the model or its behavior, as shown in [1, 3]. Indeed, in the specification of our dynamical North-South model, the two-dimensional endomorphism is defined using a demand specification (5.3.1) which amounts to requiring that the demand for industrial goods I^D is a proportion $1 - \gamma$ of *GNP*. This last specification is useful in a North-South world, because typically industrial countries consume a higher proportion of their *GNP* in the form of industrial goods, while developing countries consume proportionately more basic goods. With our specification (5.2.6) it is also possible to simulate an economy where the proportion γ depends on the *GNP* level, with γ decreasing as a function of *GNP*. We now begin a step-by-step development of our two-dimensional dynamical system. The first step will be a simple one-dimensional model.

4. One-Dimensional Models

In preparation for our main model, the two-dimensional map defined in Section 5.2, we now study a preliminary, one-dimensional model. This simple model is less realistic in economic terms than our main model of Section 5.2. Our purpose in introducing a simple model first is pedagogic: this serves to anticipate and explain the mathematical behavior of the larger model in a transparent fashion. It is important to note that the results of this paper do not depend on this simple model but rather on the main model, which is introduced and developed in Section 5.2.

We now introduce dynamics for the macroeconomic variables of the North region. The variables of the South will then be obtained as functions of those of the North, as follows.

PROPOSITION 1. *In the North-South model, the South capital is obtained from the North by the affine isomorphism,*

$$K_S = H_0 + H_1 K_N,$$

where

$$H_1 = \frac{\beta_S a_{2S} D_N}{\beta_N a_{2N} D_S}$$

and

$$H_0 = \frac{\beta_S}{D_S}\left[-\frac{a_{1N}}{a_{2N}}\, a_{2S} + a_{1S}\right] - H_1 \bar{K}_N + \bar{K}_S.$$

Proof. From (GC2.4) we have

$$K_N = \beta_N r_N + \bar{K}_N \tag{4.0.1}$$

and

$$K_S = \beta_S r_S + \bar{K}_S. \tag{4.0.2}$$

As we assume the terms of
or from (GC2.21a),

$p = (a_{1S} - r_S D_S$

or, solving for r_S,

$r_S = \frac{1}{D_S}\{(r_N L$

we now substitute (4.0.1)

$K_S = \beta_S r_S + \bar{K}$

Using (GC2.4a) to replac

$K_S = \frac{\beta_S}{D_S}\{\frac{a_{2S}}{a_2}$

and simplifying, we get t

Henceforth in Section 4,

4.1. *The Dynamics of the*

We envision a dynamic
result, after a rapid tran
values of the variables. W
of these variables. And
equation, so that $\beta < 0$.
will be defined by a func
point $\bar{x}$ subsequently), sc
Also, we write K^+ for f

$f(K) = (1 - \delta)$

where the depreciation r
small, positive values, ar

$GNP = pB + I.$

As usual, *GNP* is the in
(GC2.16).

After substitution of
mophism f may be writ

PROPOSITION 2. *The*

$f(K) = A_0 +$

:ifications without changing
ıhown in [1, 3]. Indeed, in
model, the two-dimensional
:ation (5.3.1) which amounts
ls I^D is a proportion $1 - \gamma$
North-South world, because
proportion of their *GNP* in
countries consume propor-
on (5.2.6) it is also possible
depends on the *GNP* level,
begin a step-by-step devel-
:em. The first step will be a

ensional map defined in Sec-
ensional model. This simple
r main model of Section 5.2.
t is pedagogic: this serves to
ior of the larger model in a
the results of this paper do
n the main model, which is

nomic variables of the North
btained as functions of those

he South capital is obtained

$_N + \bar{K}_S.$

(4.0.1)

(4.0.2)

As we assume the terms of trade $p = p_B$ are the same in each region, $p_S = p_N$, or from (GC2.21a),

$$p = (a_{1S} - r_S D_S)/a_{2S} = (a_{1N} - r_N D_N)/a_{2N}, \tag{4.0.3}$$

or, solving for r_S,

$$r_S = \frac{1}{D_S}\left\{(r_N D_N - a_{1N})\frac{a_{2S}}{a_{2N}} + a_{1S}\right\}, \tag{4.0.4}$$

we now substitute (4.0.1) into (4.0.2) and obtain

$$K_S = \beta_S r_S + \bar{K}_S = \frac{\beta_S}{D_S}\left\{(r_N D_N - a_{1N})\frac{a_{2S}}{a_{2N}} + a_{1S}\right\} + \bar{K}_S.$$

Using (GC2.4a) to replace r_N we have

$$K_S = \frac{\beta_S}{D_S}\left\{\frac{a_{2S} D_N}{a_{2N}}\left[\frac{K_N - \bar{K}}{\beta_N}\right] - \frac{a_{1N}}{a_{2N}} + a_{1S}\right\} + \bar{K}_S$$

and simplifying, we get the proposition. □

Henceforth in Section 4, we will write K in place of K_N, and so forth.

4.1. *The Dynamics of the One-Dimensional Model*

We envision a dynamic in which changes in the capital stock in the North result, after a rapid transit to new static equilibrium, in new equilibrium values of the variables. We use discrete dynamics to model the annual reports of these variables. And now, Equation (4.0.1) is understood as a demand equation, so that $\beta < 0$. *This differs from* [3]. The annual increments of K will be defined by a function, $f : \boldsymbol{R}\backslash\{\bar{x}\} \to \boldsymbol{R}$ (we will identify the excluded point $\bar{x}$ subsequently), so that for year $n+1$, we have $K(n+1) = f(K(n))$. Also, we write K^+ for $f(K)$. This function is assumed to be defined by

$$f(K) = (1-\delta)K + s(GNP), \quad 0 < \delta, \quad s < 1, \tag{4.1.1}$$

where the depreciation rate, δ, and the rate of savings, s, are constants with small, positive values, and

$$GNP = pB + I. \tag{4.1.2}$$

As usual, *GNP* is the inner product of goods and prices, and again, $p_I = 1$ (GC2.16).

After substitution of the expressions in the preceding section, the endomophism f may be written in the following form.

PROPOSITION 2. *The function defined in (4.1.1) may be expressed as*

$$f(K) = A_0 + A_1 K + A_2 K^2 + A_*/(K - K_0), \tag{4.1.3}$$

where the coefficients are given by

$$A_0 = (s/a)[1 + c_2\bar{K}/\beta](\bar{L} + \alpha c_2/D) - s a_2^2 c_1,$$

$$A_1 = (1-\delta) + (s/\beta)\{-\bar{K} - (c_2/a_2)(\bar{L} + \alpha c_2/D)\},$$

$$A_2 = s/\beta,$$

$$A_* = -s(c_1^2 a_2 \beta/D),$$

and the singular point ($\bar{x}$ above) is

$$K_0 = \bar{K} + a_1\beta/D.$$

4.2. *Proof of Proposition 2*

We will demonstrate the dynamical rule given above in six steps.

Step 1. First we observe:

$$p = u_1(K - K_0),$$

where $u_1 = -D/a_2\beta$, and $K_0 = \bar{K} + a_1\beta/D$.

Proof. From (GC2.21a) of Section 2 we have

$$p = (a_1 - rD)/a_2$$

and substituting for r from (GC2.4a) above,

$$p = \frac{a_1}{a_2} - \frac{(K-\bar{K})D}{a_2\beta},$$

from which we obtain

$$p = u_0 + u_1 K,$$

where u_1 is defined above, and

$$u_0 = \frac{a_1\beta + D\bar{K}}{a_2\beta}.$$

Then Step 1 follows, with

$$K_0 = -u_0/u_1 = \frac{a_1\beta + D\bar{K}}{a_2\beta} + \frac{a_2\beta}{D} = \frac{a_1\beta}{D} + \bar{K}.$$

Step 2. Continuing, we find:

$$pL = -\frac{\alpha c_2 + \bar{L}D}{a_1\beta}K + \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1).$$

Note: Combining Steps
K. Combining with Propo
the primary variables, K_N,
dimensional model.

Proof. From (GC2.3a) of

$$pL = p\left(\alpha\frac{w}{p} + \bar{L}\right)$$

and substituting for W from

$$pL = \alpha\frac{pc_2 - c_1}{D} +$$

Using Step 1,

$$pL = \left(\frac{\alpha c_2}{D} + \bar{L}\right)$$

$$= -\left(\frac{\alpha c_2}{D} + \bar{L}\right.$$

$$= -\frac{\alpha c_2 + \bar{L}L}{\beta a_1}$$

$$= -\frac{\alpha c_2 + \bar{L}L}{\beta a_1}$$

$$= -\frac{\alpha c_2 + \bar{L}L}{\beta a_1}$$

completing the derivation.

Step 3. Next, see that:

$$pK = -\frac{D}{a_2\beta}K^2 +$$

Proof. From Step 1 we h

$$pK = p_1(K - K_0$$

$$= p_1K^2 - p_1$$

$$= -\frac{D}{a_2\beta}K^2 -$$

$$= -\frac{D}{a_2\beta}K^2 -$$

Step 4. Putting these togeth

Note: Combining Steps 1 and 2, we have expressed L as a function of K. Combining with Proposition 1, we see that the evolution of all four of the primary variables, K_N, L_N, K_S and L_S, are determined from our one-dimensional model.

Proof. From (GC2.3a) of Section 2 we have

$$pL = p\left(\alpha\frac{w}{p} + \bar{L}\right) = \alpha w + p\bar{L}$$

and substituting for W from (GC2.21b),

$$pL = \alpha\frac{pc_2 - c_1}{D} + p\bar{L} = \left(\frac{\alpha c_2}{D} + \bar{L}\right)p - \frac{\alpha c_1}{D}.$$

Using Step 1,

$$\begin{aligned} pL &= \left(\frac{\alpha c_2}{D} + \bar{L}\right)p_1(K - K_0) - \frac{\alpha c_1}{D} \\ &= -\left(\frac{\alpha c_2}{D} + \bar{L}\right)\frac{D}{a_1\beta}K + \left(\frac{\alpha c_2}{D} + \bar{L}\right)\frac{D}{a_1\beta}K_0 - \frac{\alpha c_1}{D} \\ &= -\frac{\alpha c_2 + \bar{L}D}{\beta a_1}K + \frac{\alpha c_2 + \bar{L}D}{\beta a_1}(\bar{K} + \frac{a_1\beta}{D}) - \frac{\alpha c_1}{D} \\ &= -\frac{\alpha c_2 + \bar{L}D}{\beta a_1}\,K + \frac{\alpha c_2 + \bar{L}D}{\beta a_1}\,\bar{K} + \frac{\alpha c_2}{D} + \bar{L} - \frac{\alpha c_1}{D} \\ &= -\frac{\alpha c_2 + \bar{L}D}{\beta a_1}\,K + \frac{\alpha c_2 + \bar{L}D}{\beta a_1}\,\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1), \end{aligned}$$

completing the derivation.

Step 3. Next, see that:

$$pK = -\frac{D}{a_2\beta}\,K^2 + \left[\frac{D}{a_2\beta}\,\bar{K} + \frac{a_1}{a_2}\right]K.$$

Proof. From Step 1 we have

$$\begin{aligned} pK &= p_1(K - K_0)K \\ &= p_1K^2 - p_1K_0K \\ &= -\frac{D}{a_2\beta}\,K^2 + \frac{D}{a_2\beta}\left[\bar{K} + \frac{a_1\beta}{D}\right]K \\ &= -\frac{D}{a_2\beta}\,K^2 + \left[\frac{D}{a_2\beta}\bar{K} + \frac{a_1}{a_2}\right]K. \end{aligned}$$

Step 4. Putting these together, we have

$$pB = C_0 + C_1 K + C_2 K^2,$$

where

$$\begin{aligned}
C_0 &= \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1), \\
C_1 &= -\frac{\alpha c_2^2}{a_1\beta D} - \frac{c_2\bar{L}}{a_1\beta} - \frac{\bar{K}}{\beta} - \frac{a_1}{D}, \\
C_2 &= 1/\beta.
\end{aligned}$$

Proof. From Section 2 (GC2.20a) we have

$$\begin{aligned}
pB &= p\frac{c_2 L - a_2 K}{D} \\
&= \frac{c_2}{D}pL - \frac{a_2}{D}pK,
\end{aligned}$$

in which we may replace pL with Step 2, and pK by Step 3, obtaining

$$\begin{aligned}
pB &= \frac{c_2}{D}\left\{-\frac{\alpha c_2 + \bar{L}D}{a_1\beta}K + \frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1)\right\} \\
&\quad - \frac{a_2}{D}\left\{-\frac{D}{a_2\beta}K^2 + \left[\frac{D}{a_2\beta}\bar{K} + \frac{a_1}{a_2}\right]K\right\} \\
&= \frac{1}{\beta}K^2 - \left\{\frac{\alpha c_2^2}{a_1\beta D} + \frac{c_2\bar{L}}{a_1\beta} + \frac{\bar{K}}{\beta} + \frac{a_1}{D}\right\}K \\
&\quad + \left\{\frac{\alpha c_2 + \bar{L}D}{a_1\beta}\bar{K} + \bar{L} + \frac{\alpha}{D}(c_2 - c_1)\right\},
\end{aligned}$$

which is Step 4.

Step 5. Similarly, see that:

$$I = I_0 + I_1 K + I_*/(K - K_0),$$

where

$$\begin{aligned}
I_0 &= -\left[\frac{c_1\bar{L}}{D} + \frac{\alpha c_1 c_2}{D^2}\right], \\
I_1 &= \frac{a_1}{D}, \\
I_* &= -\frac{\alpha\beta a_2 c_1^2}{D^3}.
\end{aligned}$$

Proof. From Section 2 (C

$I = \frac{a_1 K - c_1 L}{D}$

which is Step 5

$GNP = G_0 + G_1 K$

where

$G_0 = C_0 + I_0 =$

$G_1 = C_1 + I_1 = -$

$G_2 = C_2 = 1/\beta,$

$G_* = I_* = -\alpha\beta a_2 c_1^2$

Proof. From Section 4 (4

$GNP = pB + I,$

in which we may replace

$GNP = C_2K^2 + ($

which completes our deriva

4.3. *Preliminaries on Quad*

In the preceding sections we
generating a semi-cascade (
the North-South model. Thi

Proof. From Section 2 (GC2.20b) we have

$$\begin{aligned}
I &= \frac{a_1 K - c_1 L}{D} \\
&= \frac{a_1}{D} K - \frac{c_1}{D}[\frac{\alpha w}{p} + \bar{L}] \\
&= \frac{a_1 K - c_1 \bar{L}}{D} - \frac{\alpha c_1}{D} \frac{w}{p} \\
&= \frac{a_1 K - c_1 \bar{L}}{D} - \frac{\alpha c_1}{D^2} \left[c_2 - \frac{c_1}{p} \right] \\
&= \frac{a_1 K - c_1 \bar{L}}{D} - \frac{\alpha c_1 c_2}{D^2} + \frac{\alpha c_1^2}{D^2} \frac{1}{p} \\
&= \frac{a_1}{D} K - \left[\frac{c_1 \bar{L}}{D} + \frac{\alpha c_1 c_2}{D^2} \right] + \frac{\alpha c_1^2}{D^2} \frac{1}{p_1 (K - K_0)} ,
\end{aligned}$$

which is Step 5

$$GNP = G_0 + G_1 K + G_2 K^2 + G_* / (K - K_0),$$

where

$$\begin{aligned}
G_0 &= C_0 + I_0 = \frac{\alpha c_2 + \bar{L} D}{a_1 \beta} \bar{K} + (1 - \frac{c_1}{D}) \bar{L} + \frac{\alpha}{D}(c_2 - c_1) - \frac{\alpha c_1 c_2}{D^2} \\
G_1 &= C_1 + I_1 = - \left[\frac{\alpha c_2^2}{a_1 \beta D} + \frac{c_2 \bar{L}}{a_1 \beta} + \frac{\bar{K}}{\beta} \right] \\
G_2 &= C_2 = 1/\beta, \\
G_* &= I_* = -\alpha \beta a_2 c_1^2 / D^3.
\end{aligned}$$

Proof. From Section 4 (4.1.2) we have

$$GNP = pB + I,$$

in which we may replace pB by Step 4, and I by Step 5, obtaining

$$GNP = C_2 K^2 + (C_1 + I_1) K + (C_0 + I_0) + I_* \frac{1}{K - K_0} .$$

which completes our derivation. □

4.3. *Preliminaries on Quadratic Maps*

In the preceding sections we have obtained an endomorphism of real numbers, generating a semi-cascade (discrete dynamical system), for the dynamics of the North-South model. This one-dimensional model will be useful to us, as we will see later in the study of our main (two-dimensional) model. This is because dynamics in one dimension has been extensively studies, whereas dynamics in two dimension is a current frontier. To relate this one-dimensional model to the well known logistic map, we will make use of the following.

PROPOSITION 3. *A quadratic function, $f : \Re \to \Re$, defined by*

$$f(x) = A_0 + A_1 x + A_2 x^2$$

with $A_2 \neq 0$, and the discriminant $\Delta = (A_1 - 1)^2 - 4A_0A_2 > 0$, has a repelling fixed point at

$$B_0 = -\frac{(A_1 - 1)}{2A_2} + \frac{\Delta}{2A_2}$$

with its distinct preimage at $B_0 + B_1$, where

$$B_1 = -A_1/A_2 - 2B_0.$$

The affine function

$$x : \Re \to \Re;\ y \mapsto x(y) = B_0 + B_1 y$$

is an affine isomorphism, and conjugates f into the canonical form for the quadratic family

$$g(y) = x^{-1}(f(x(y))) = \mu y(1 - y),$$

with

$$\mu = 1 + \Delta.$$

Furthermore, the usual domain of this logistic function, $y \in J = [0, 1]$, is mapped to an interval $x \in I = [B_0, B_0 + B_1]$, in the orientation preserving case $B_0 > 0$, else $x \in I = [B_0 + B_1, B_0]$, by this affine isomorphism.

Proof. To compute the next value of y under the conjugate map, we apply the inverse map to y^+,

$$\begin{aligned} y^+ &= -\frac{B_0}{B_1} + \frac{1}{B_1} x^+ \\ &= -\frac{B_0}{B_1} + \frac{1}{B_1} f(x) \\ &= -\frac{B_0}{B_1} + \frac{1}{B_1}[A_0 + A_1 x + A_2 x^2], \end{aligned}$$

and then with $x \mapsto y$,

$$\begin{aligned} y^+ &= -\frac{B_0}{B_1} + \frac{A_0}{B_1} + \frac{A_1}{B_1}(B_0 + B_1 y) + \frac{A_2}{B_1}(B_0 + B_1 y)^2 \\ &= \left[-\frac{B_0}{B_1} + \frac{A_0}{B_1} + \frac{A_1}{B_1} B_0 + \frac{A_2}{B_1} B_0^2\right] \\ &\quad + [A_1 + 2A_2 B_0]y + (A_2 B_1)y^2. \end{aligned}$$

Now we equate this with th

$$y^+ = g(y) = \mu y(1$$

term by term.

For degree zero,

$$-\frac{B_0}{B_1} + \frac{A_0}{B_1} + \frac{A_1}{B_1}$$

and as $A_2 \neq 0$ and $B_1 \neq 0$

$$A_2 B_0^2 + (A_1 - 1)$$

from which, by the binomi

$$B_0 = -\frac{(A_1 - 1 \pm}{2A_2}$$

Note: The quadratic equ of the map f, so the $\pm$ yiel two possible values for B_0

$$f'(B_0) = A_1 + 2A$$

we choose the positive sig other root, with the minus a fold bifurcation, and initi distinct preimage is $B_0^- +$ the critical point is $x_e = -$

For degree one,

$$\mu = A_1 + 2A_2 B_0$$

and for degree two,

$$\mu = -A_2 B_1.$$

Subtracting these two expr

$$B_1 = -\frac{A_1}{A_2} - 2B$$

completing the specificatio sion for μ above we obtain

COROLLARY 4. *Given th*

$$f(x) = A_0 + A_1 x$$

with $A_2 \neq 0$, and $(A_1 - 1$

$$B_0 = -\frac{A_1 - 1 +}{2A_2}$$

Now we equate this with the desired canonical form,

$$y^+ = g(y) = \mu y(1-y) = 0 + \mu y + (-\mu)y^2$$

term by term.

For degree zero,

$$-\frac{B_0}{B_1} + \frac{A_0}{B_1} + \frac{A_1}{B_1}B_0 + \frac{A_2}{B_1}B_0^2 = 0,$$

and as $A_2 \neq 0$ and $B_1 \neq 0$

$$A_2 B_0^2 + (A_1 - 1)B_0 + A_0 = 0$$

from which, by the binomial formula,

$$B_0 = -\frac{(A_1 - 1 \pm \Delta)}{2A_2}\,.$$

Note: The quadratic equation for B_0 here is the condition for a fixed point of the map f, so the $\pm$ yields the two fixed points. As the slope of f at these two possible values for B_0 is

$$f'(B_0) = A_1 + 2A_2 B_0 = 1 \pm \Delta$$

we choose the positive sign for the repelling fixed point. If B_0^- denotes the other root, with the minus sign, then this is the paired fixed point, created by a fold bifurcation, and initially attractive, for Δ small and positive. Then its distinct preimage is $B_0^- + B_1^-$, where $B_1^- = -A_1/A_2 - 2B_0^-$. Also, note that the critical point is $x_e = -A_1/2A_2$.

For degree one,

$$\mu = A_1 + 2A_2 B_0$$

and for degree two,

$$\mu = -A_2 B_1.$$

Subtracting these two expressions and solving for B_1,

$$B_1 = -\frac{A_1}{A_2} - 2B_0$$

completing the specification of the affine isomorphism. From the first expression for μ above we obtain its form in the proposition. □

COROLLARY 4. *Given the function* $f : \Re\backslash\{\bar{x}\} \to \Re$*, defined by*

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_*/(x - \bar{x})$$

with $A_2 \neq 0$*, and* $(A_1 - 1)^2 > 4A_0A_2$*, then* $y \mapsto x = B_0 + B_1 y$ *with*

$$B_0 = -\frac{A_1 - 1 + \Delta}{2A_2}$$

and

$$B_1 = -A_1/A_2 - 2B_0$$

is an affine isomorphism, and conjugates f to the canonical form $g : \Re\backslash\{\bar{y}\} \to \Re$, *with*

$$g(y) = x^{-1}(f(x(y))) = \mu y(1-y) + \nu/(y-\bar{y})$$

with $\nu = A_*/B_1^2$, $\bar{y} = \bar{x}/B_1 - B_0/B_1 = x^{-1}(\bar{x})$, *and* $\mu + \Delta$ *as above. And as above, the usual domain of the logistic function,* $y \in J = [0, 1]$, *assuming* $\bar{y} \notin J$, *is again mapped to the interval,* $x \in I = [B_0, B_0 + B_1]$, *by the affine isomorphism.*

Proof. The quadratic terms are conjugated as shown, according to Proposition 3 above. For the last term, see that

$$(1/B_1)\frac{A_*}{x-\bar{x}} = \frac{\nu}{y-\bar{y}}$$

with which, the formula for g is obtained. □

Remark. If the singular point lies outside the interval J, then this interval as approximately the invariant interval defined by the initially repelling fixed point and its distinct preimage. In case the point $\bar{y}$ lies to the right of the interval J, the domain of g should be reduced to the subinterval J^* defined by the expanding fixed point and its nearby preimage. In case $\bar{y}$ lies to the left of J, then the interval may be increased to J^*. The case with $\bar{y}$ in the interval shown in Figure 2.

The invariant interval of g, J^*, is not identical to the reference interval, $J = [0, 1]$ unless $\nu = 0$. Likewise, we have an interval for f, I^*, not identical to the corresponding reference interval, $I = [B_0, B_0 + B_1]$.

In summary, we see that in the case in which the singular point is outside the interval of interest, our one-dimensional model must behave exactly like the well-known logistic (or quadratic) map, with a convergent sequences of period-doubling bifurcations, and chaotic attractors. In the other case (which occurs with reasonable values of our numerous parameters) the behavior should be similar. This is difficult (but possible) to establish analytically, but we will use simulation instead.

4.4. *Simulations*

We now establish that, indeed, the behavior of our one-dimensional model is that of the familiar logistic function, even though the singularity falls in the domain of the map. We begin by fixing values for the many parameters appearing in this dynamical system. First, let $\delta = 0.1$ and $s = 0.08$. For the

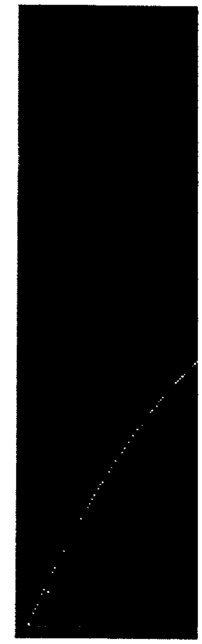

Figure 2a. Graph of the one-dime
at the left. This is the singularity, s

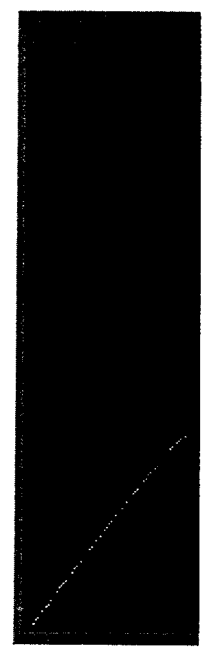

Figure 2b. Graph of the one-dime
in the map.

canonical form $g : \Re \backslash \{\bar{y}\} \to$

$\nu/(y - \bar{y})$

$\bar{x}$), *and* $\mu + \Delta$ *as above. And*
ion, $y \in J = [0, 1]$, *assuming*
$= [B_0, B_0 + B_1]$, *by the affine*

.s shown, according to Propo-

□

e interval J, then this interval
by the initially repelling fixed
oint $\bar{y}$ lies to the right of the
to the subinterval J^* defined
image. In case $\bar{y}$ lies to the left
The case with $\bar{y}$ in the interval

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$B_0, B_0 + B_1]$.
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ctors. In the other case (which
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e) to establish analytically, but

of our one-dimensional model
though the singularity falls in
alues for the many parameters
$\delta = 0.1$ and $s = 0.08$. For the

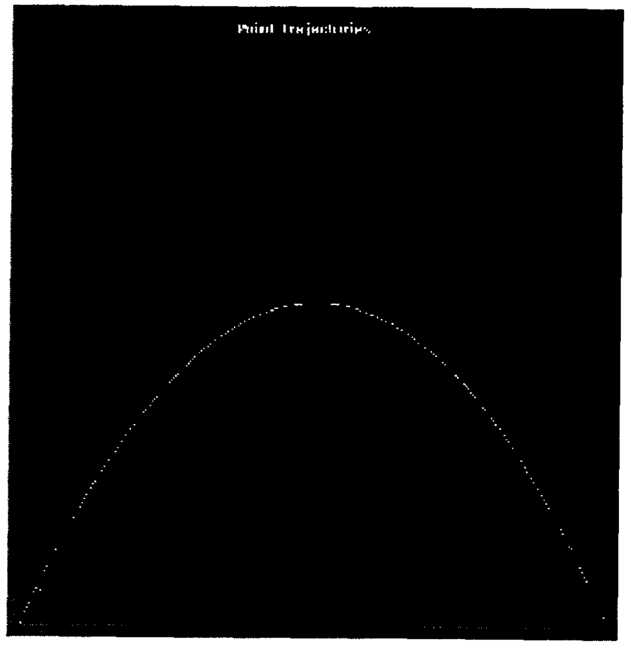


Figure 2a. Graph of the one-dimensional model with $0 < x < 163$. Note the gap in the graph at the left. This is the singularity, shown enlarged in Figure 3.

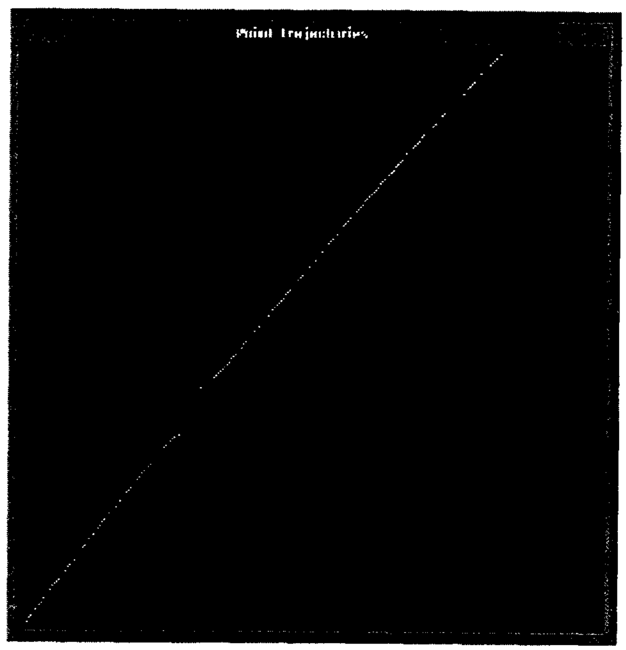


Figure 2b. Graph of the one-dimensional model with $0 < x < 20$, illustrating the singularity in the map.

others, our guide will be table (c) on page 44 of [3], except for the sign of β which we reverse. Thus, in the North,

$$\begin{aligned} a_1 &= 2, & \bar{K} &= 12, \\ a_2 &= 0.15, & \bar{L} &= 0.5, \\ c_1 &= 1.8, & \alpha &= 6, \\ c_2 &= 1.7, & \beta &= -9.7. \end{aligned}$$

These are chosen so that $p, r, w, L, K, B, I > 0$ in each region. Note that the control parameter μ in the transformed dynamical system depends upon all of these values. The derived constants are then approximately:

$$D = 3.13$$

$$\begin{aligned} A_0 &= -0.058524, & B_0 &= 0.167727, \\ A_1 &= 1.350306, & B_0^- &= 42.306847, \\ A_2 &= -0.008247, & B_1^- &= 79.110881, \\ A_* &= 0.1201, & B_1 &= 163.389119, \end{aligned}$$

with the singularity at $\bar{x} = 5.801917$ and the attracting fixed point at 42.316339, see Figures 2a and 2b.

The response diagram for function f of (4.1.1) – with all the parameters fixed with these values except for α, which is regarded as the control parameter in the simulation – is the familiar orbit diagram for the quadratic family, as shown in Figure 3.

5. Two-Dimensional Models

In the first dynamical system studied above, we had an evolution in the North variables, while the South variables were to be determined from their Northern siblings by an algebraic relation. We now want to consider a more symmetric dynamic, in which the corresponding variables in both regions are in mutual coevolution.

5.1. *A Preliminary Model*

Here we rewrite the one-dimensional model as a two-dimensional model without changing the dynamics for K_N. That is, instead of obtaining K_S from K_N after each timestep by conjugation with the affine isomorphism of Proposition 1, which assumed a rapid settling to static equilibrium, we will derive a semi-cascade for K_S parallel to that of K_N.

From Proposition 1 we have

$$K_S = H_0 + H_1 K_N, \tag{5.1}$$

while from Proposition 2,

$$K_N(n+1) = f(K_N(n)),$$

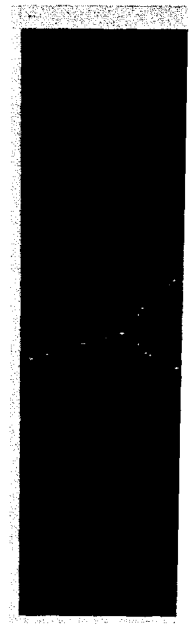


Figure 3. Response diagram
familiar figure for the quadratic
dynamical system. Each value
a particular map generating th
attractor of the dynamical syste
(as in equilibrium theory), peri
in economic data).

or writing f_N in place of

$$K_N^+ = f_N(K_N).$$

Note that the inverse of P

$$K_N = \frac{K_S - H_0}{H_1}$$

We now apply the map
(5.3) to the rights-hand si
ing result.

PROPOSITION 5. *The d*
for K_S, which may be ex

$$K_S(n+1) = f_S$$

where the generating end

$$f_S(y) = A_{0S} + $$

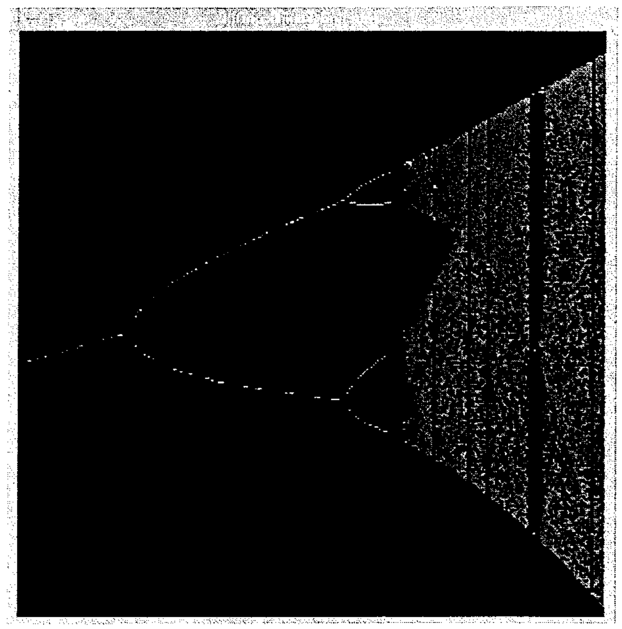


Figure 3. Response diagram for the one-dimensional model with $6 < \alpha < 8$. This is the familiar figure for the quadratic family. The vertical axis is the domain of the one-dimensional dynamical system. Each value of the control parameter α determines a vertical interval, and a particular map generating the dynamic. The white point (or set of points) is the unique attractor of the dynamical system for the given value of the control parameter: a point attractor (as in equilibrium theory), periodic attractor (as in business cycles), or a chaotic attractor (as in economic data).

or writing f_N in place of f,

$$K_N^+ = f_N(K_N). \tag{5.2}$$

Note that the inverse of Proposition 1 is

$$K_N = \frac{K_S - H_0}{H_1}\,. \tag{5.3}$$

We now apply the map of (5.1) to the left-hand side of (5.2), and its inverse (5.3) to the rights-hand side, as in the proof of Proposition 3, with the following result.

PROPOSITION 5. *The dynamic (5.2) for K_N implies a conjugate dynamic for K_S, which may be expressed,*

$$K_S(n+1) = f_S(K_S(n)) \quad or \quad K_S^+ = f_S(K_S),$$

where the generating endomorphism is

$$f_S(y) = A_{0S} + A_{1S}y + A_{2S}y^2 + A_{*S}\frac{1}{y - \bar{y}}$$

and the coefficients are given by

$$
\begin{aligned}
A_{0S} &= H_0 + H_1 A_0 - A_1 H_0 + A_2 \frac{H_0^2}{H_1}, \\
A_{1S} &= A_1 - 2A_2 \frac{H_0}{H_1}, \\
A_{2S} &= \frac{A_2}{H_1}, \\
A_{*S} &= H_1^2 A_*, \\
\bar{y} &= H_0 + K_N^0 H_1.
\end{aligned}
$$

Note: Given K_S and all the parameters, we obtain all the variables. But, we will use different values for the parameters in the South: again, as in Section 4.4, we let $\delta = 0.1$ and $s = 0.08$. For the others, we again refer to table (c) on page 44 of [3], except for the sign of β which we reverse. Thus, in the South,

$$
\begin{aligned}
&a_1 = 4.5, \quad \bar{K} = 2.7, \\
&a_2 = 0.02, \quad \bar{L} = -2, \\
&c_1 = 0.01, \quad \alpha = 75, \\
&c_2 = 3, \qquad \beta = -0.025.
\end{aligned}
$$

These are chosen so that $p, r, w, L, K, B, I > 0$ in each region. Note that the control parameter μ in the transformed dynamical system depends upon all of these values. The derived constants are then approximately:

$$
\begin{aligned}
&D = 13.5 \\
&A_0 = -750.642844, \quad B_0 = 2.691218, \\
&A_1 = 558.498719, \quad B_0^- = 2.694575, \\
&A_2 = -103.512843, \quad B_1^- = 0.006303, \\
&A_* = 0.0000000008, \quad B_1 = 0.013018,
\end{aligned}
$$

with the singularity at $\bar{x} = 2.691667$ and the attracting fixed point at 2.694576.

Proof. From Proposition 1 we have

$$K_S = H_0 + H_1 K_N$$

with inverse

$$K_N = \frac{K_S - H_0}{H_1},$$

while from Proposition 2,

$$K_N(n+1) = f_N(K_N(n)).$$

As in the proof of Proposi
its inverse to this equation, g

$$
\begin{aligned}
K_S^+ &= H_0 + H_1 K \\
&= H_0 + H_1 f(\\
&= H_0 + H_1 f \Big(\\
&= H_0 + H_1 A_0 \\
&\quad + H_1 A_* \frac{}{(K} \\
&= H_0 + H_1 A_0 \\
&\quad + H_1^2 A_* \frac{}{K_S}
\end{aligned}
$$

from which the proposition fo

We may apply the Corollary
dynamical systems (4.2) and (
logistic endomorphism,

$$k_N^* = \mu_N k_N (1 - k_N$$

$$k_S^* = \mu_S k_S (1 - k_S) -$$

both on the unit interval, with

$$\mu_N = 1 + \sqrt{(A_{1N} -}$$

$$\mu_S = 1 + \sqrt{(A_{1S} - 1}$$

$$\nu_N = A_{*N} / B_{1N}^2,$$

$$\nu_S = A_{*S} / B_{1S}^2.$$

That is, we have in this mo
logistic maps, each of the form

$$f(K) = (1 - \delta) K + s$$

or equivalently,

$$f(K) = (1 - \delta) K + s$$

We now seek to couple them th

As in the proof of Proposition 3, we now apply the affine isomorphism and its inverse to this equation, getting

$$
\begin{aligned}
K_S^{+} &= H_0 + H_1 K_N^{+} \\
&= H_0 + H_1 f(K_N) \\
&= H_0 + H_1 f\left(\frac{K_S - H_0}{H_1}\right) \\
&= H_0 + H_1 A_0 + H_1 A_1 \frac{K_S - H_0}{H_1} + H_1 A_2 \left[\frac{K_S - H_0}{H_1}\right]^2 \\
&\quad + H_1 A_* \frac{1}{(K_S - H_0)/H_1 - K_0} \\
&= H_0 + H_1 A_0 + A_1 (K_S - H_0) + \frac{A_2}{H_1}\left(K_S^2 - 2H_0 K_S + H_0^2\right) \\
&\quad + H_1^2 A_* \frac{1}{K_S - H_0 - K_0 H_1}
\end{aligned}
$$

from which the proposition follows. □

We may apply the Corollary of Proposition 3 independently to each of the dynamical systems (4.2) and (5.1), obtaining the (uncoupled) two-dimensional logistic endomorphism,

$$
\begin{aligned}
k_N^* &= \mu_N k_N (1 - k_N) + \nu_N/(k_N - k_{N0}), \\
k_S^* &= \mu_S k_S (1 - k_S) + \nu_S/(k_S - k_{S0}),
\end{aligned}
$$

both on the unit interval, with

$$
\begin{aligned}
\mu_N &= 1 + \sqrt{(A_{1N} - 1)^2 - 4A_{0N}A_{2N}} = 1 + \Delta_N, \\
\mu_S &= 1 + \sqrt{(A_{1S} - 1)^2 - 4A_{0S}A_{2S}} = 1 + \Delta_S, \\
\nu_N &= A_{*N}/B_{1N}^2, \\
\nu_S &= A_{*S}/B_{1S}^2.
\end{aligned}
$$

That is, we have in this model a minor modification of two (uncoupled) logistic maps, each of the form

$$f(K) = (1 - \delta)K + s(GNP),$$

or equivalently,

$$f(K) = (1 - \delta)K + s(pB + I).$$

We now seek to couple them through p.

5.2. *The Main Model*

We will work with an endomorphism of the plane

$$T : \mathbf{R}^2 \to \mathbf{R}^2; (K_N, K_S) \mapsto (K_N^+, K_S^+)$$

defined as in the one-dimensional model by

$$K_N^+ = s_N(pB_N + I_N) + (1 - \delta_N)K_N, \tag{5.2.1}$$

$$K_S^+ = s_S(pB_S + I_S) + (1 - \delta_S)K_S, \tag{5.2.2}$$

where the terms of trade, p, are the same in both regions, because markets are competitive. These equations predict growth of capital stock in one fiscal period. As before, $pB + I$ is the *GNP* (gross national product), s is the savings rate, and δ is depreciation. In our simulations, we will use $s \approx 12/100$, and $\delta \approx 10/100$, and for both regions.

The time evolution of all of the variables in each system is to be found by the iteration of the mapping T, beginning with any initial state, (K_N^0, K_S^0). To complete the definition of the endomorphism T and thus the dynamics of the model, we explain the determination of the intermediate variables, p, B, I, in each region. These are determined by equation (GC2.22) of [3] modified as follows:

$$\beta_N = 0, \quad \bar{K}_N = K_N; \quad \beta_S = 0, \quad \bar{K}_S = K_S.$$

We recall, from [3], the equation

$$A_T p^2 + (C_T + I_T^D)p - V_T = 0, \tag{GC2.22}$$

where here $A = \beta a_1 a_2 / D^2$, and C and V are defined below. Equation (GC2.22) then becomes, with $\beta = 0$ in each region,

$$(C_T + I_T^D)p - V_T = 0, \tag{5.2.3}$$

using the convention of Section 2. Here, the symbolic expressions C, V and I^D, are defined by

$$C = (1/D)[c_1\bar{L} - a_1 K + \alpha c_1 c_2 / D], \tag{5.2.4}$$

$$V = \alpha c_1^2 / D^2, \tag{5.2.5}$$

$$I^D = GNP(1 - \gamma), \tag{5.2.6}$$

where $\gamma \in (0, 1)$. In fact, we will choose $\gamma \approx 60/100$. In any case, we would like $s + (1 - \gamma) \ll 1$. Note that C is a function of K in each region, V is a constant, and *GNP* in the expression for I^D is to be determined from the formula $GNP = pB + I$. Equation (5.2.6) is the assumption that demand for industrial goods is proportional to *GNP*, as described above, in each region. This treats the two goods, B and I, symmetrically. Note that the values of B and I are directly computed as function of K (in each region) by

Equations (3.1) and (3.2), but
available. We obtain this valu
in the static model as describe

Once p is determined, we
tion (4.1.2), and equations (G
as:

$$GNP = p(c_2 L - a_2 K$$
$$= p(\alpha c_2^2 / D^2 +$$
$$+ [-2\alpha c_1 c_2$$

for each region. Note that Eq
but our expression (5.2.7) abov
we obtain a quadratic equation

We begin by rewriting (5.2.

$$p[C_T + (1 - \gamma)GNP]$$

and using (5.2.7), this yields

$$E_T p^2 + (C_T + F_T)p$$

where

$$E = (1 - \gamma)[\alpha c_2^2 / D^2$$

$$F = (1 - \gamma)[-2\alpha c_1 c_2$$

and

$$G = (1 - \gamma)\alpha c_1^2 / D^2.$$

Thus, computing L from K
quadratic equation (5.2.9) are k
two real roots, we choose the la
(5.2.7) we have *GNP* in each r
complete.

An interesting simplification
ing rK for *GNP* in the dynam
tions (5.2.1) and (5.2.2). This thi
and we may return to it in a futu

5.3. *Simulation Results*

For the first two-dimensional m
ure 4. Throughout this section, t
Section 4.4 (for the North) and S
the figure captions.

Equations (3.1) and (3.2), but the value of p in this expression is not directly available. We obtain this value, assuming the rapid approach to equilibrium in the static model as described in Section 1, as described below.

Once p is determined, we obtain the *GNP*, which is given by Equation (4.1.2), and equations (GC2.20a,b), (GC2.21a), and (GC2.3) from [3], as:

$$\begin{aligned} GNP &= p(c_2L - a_2K)/D + (a_1K - c_1L)/D \\ &= p(\alpha c_2^2/D^2 + c_2\bar{L}/D - a_2K/D] \\ &\quad + [-2\alpha c_1c_2/D^2 + a_1K/D - c_1\bar{L}/D] + \alpha c_1^2/D^2p \end{aligned} \tag{5.2.7}$$

for each region. Note that Equation (5.2.3) determines p if *GNP* is known, but our expression (5.2.7) above requires p. When this circularity is resolved, we obtain a quadratic equation for p with all coefficients known.

We begin by rewriting (5.2.3), using (5.2.6), in the form

$$p[C_T + (1-\gamma)GNP] - V_T = 0, \tag{5.2.8}$$

and using (5.2.7), this yields

$$E_Tp^2 + (C_T + F_T)p + (G_T - V_T) = 0, \tag{5.2.9}$$

where

$$E = (1-\gamma)[\alpha c_2^2/D^2 - (a_2K + c_2\bar{L})/D],$$

$$F = (1-\gamma)[-2\alpha c_1c_2/D^2 + a_1K/D - c_1\bar{L}/D],$$

and

$$G = (1-\gamma)\alpha c_1^2/D^2.$$

Thus, computing L from K in each region, all the coefficients of the quadratic equation (5.2.9) are known. We solve this equation, and in case of two real roots, we choose the larger one for the current value of p. Then from (5.2.7) we have *GNP* in each region, and the specification of the map T is complete.

An interesting simplification to our main model results from substituting rK for *GNP* in the dynamical rules for the 2D endomorphism, Equations (5.2.1) and (5.2.2). This third model has been studied by Di Matteo [10] and we may return to it in a future publication.

5.3. *Simulation Results*

For the first two-dimensional model, the response diagram is shown in Figure 4. Throughout this section, the values of all the constants are as given in Section 4.4 (for the North) and Section 5.1 (for the South) except as noted in the figure captions.

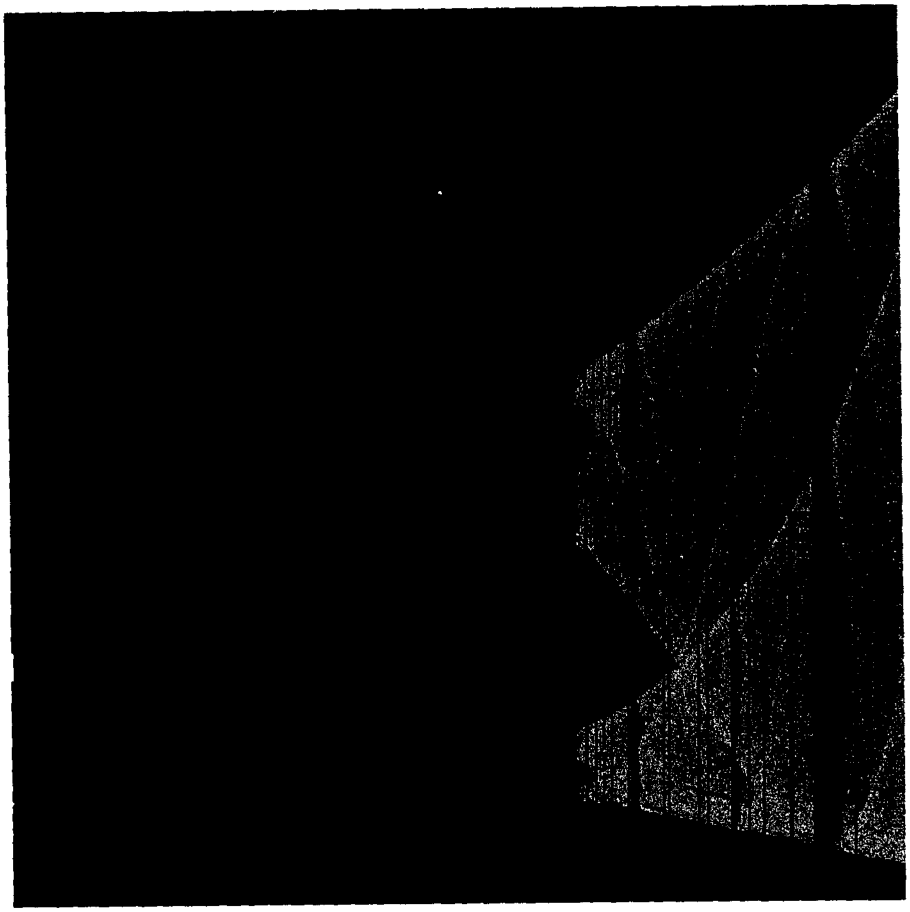
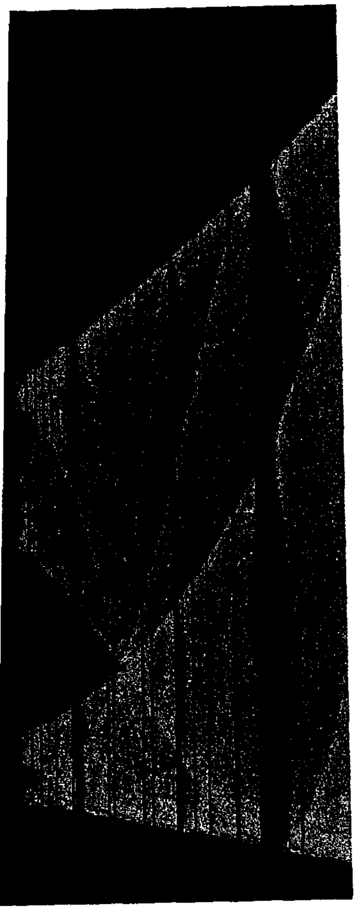


Figure 4. Response diagram for the first of the two-dimensional models. Here we vary α_N from 31 to 49 while holding α_S fixed at 20. The horizontal axis represents the control parameter, α_N, while the vertical axis represents the North capital supply, K_N, after several iterations. The interpretation of this diagram is identical to that of Figure 3, except that here the vertical axis is the one-dimensional projection of a two-dimensional state space.

It is here that our experience with the one-dimensional model is pedagogically useful, as we see a strong similarity in the response diagrams. In this case, we have a two-dimensional state space, of the variables K_N and K_S, and a one-dimensional control space, of the control parameter, α_N. Thus, the response diagram is three-dimensional. But here we have reduced it to a two-dimensional graphic by projection. The vertical axis represents the two-dimensional state space (of the capital stocks in North only), and the horizontal axis is the control space of the environmental variable α_N. As the two equations of the first two-dimensional model are uncoupled, this projection gives us exactly the response diagram of the one-dimensional model



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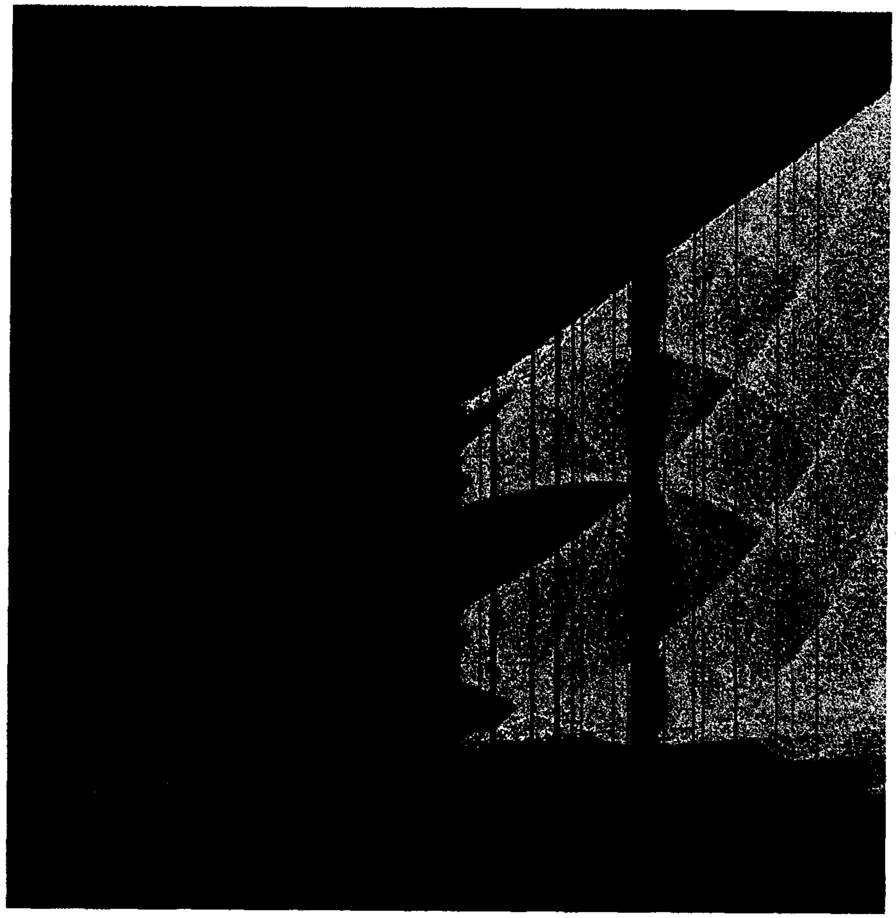

Figure 5. Response diagram for the second of the two-dimensional models. Here we vary the α_S from 40 to 90 while holding the α_N fixed at 6. The horizontal axis represents the control parameter, α_S. Both North and South capitol stocks are plotted on the vertical axis. This view of the response diagram is constructed as follows. For each value of the control parameter (horizontal axis) there corresponds a dynamical system on the state space, a rectangle in the plane of the state variables K_N and K_S. This discrete dynamical system has a single attractor, either a point (static attractor), a finite point set of $k > 1$ points (a k-periodic attractor), or an infinite set (chaotic attractor). In any case, we (step 1) project this attractor onto the K_N axis, then (step 2) project this attractor onto the K_S axis, and then (step 3) superimpose both projections onto the same interval of real numbers. Finally (step 4), this picture is inserted into the response diagram as a vertical line segment over the chosen value of the control parameter. Note that there are two figures, similar to Figure 4, which are superimposed here, one for each of the projections: K_N and K_S.

studied above, that is, Figure 3. (Some of the parameters differ, however, between these two figures.) We see, at the left of the response diagram, a period doubling bifurcation, followed by the familiar convergent sequences

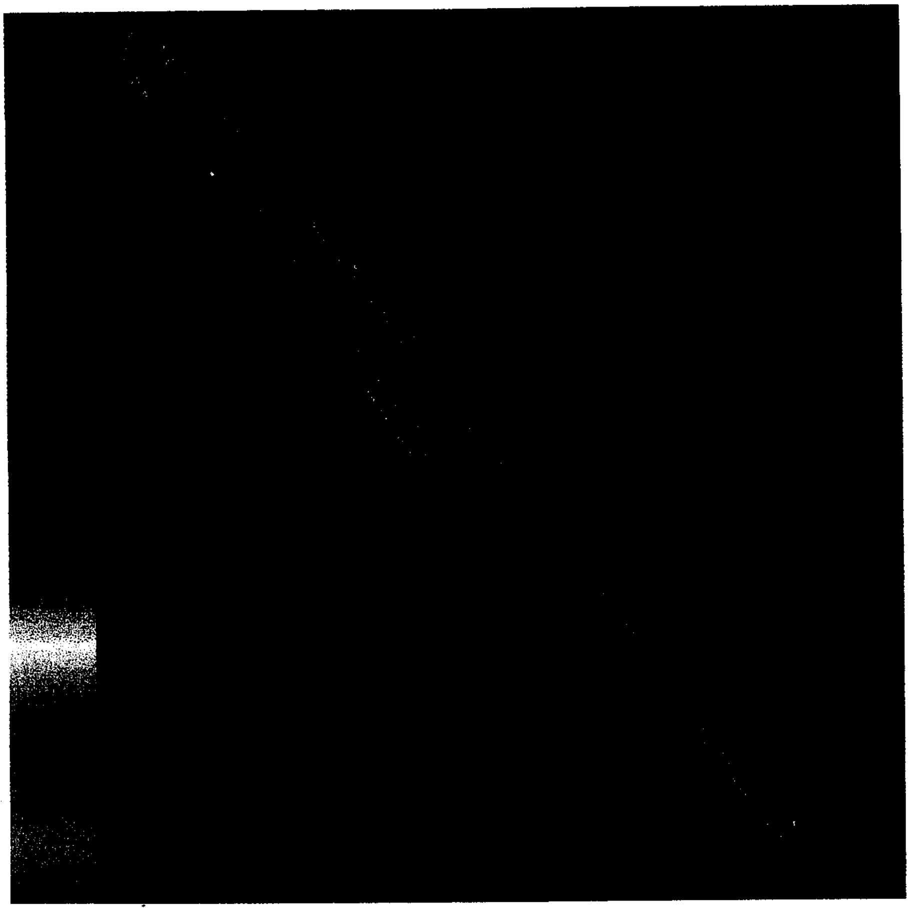

Figure 6. A histogram of the attractor in the two-dimensional state space of K_N and K_S, for a particular value of the control parameter, $\alpha_S = 80$. The horizontal axis represents values of K_N, the vertical, K_S. The bar on the lower left shows the gray scale code, from black (no points of the trajectory in a unit area) to white (maximum number of trajectory points in a unit area).

of similar events. As we see this in projection, we may understand that there is a periodic attractor in the two-dimensional state space of the variables K_N and K_S, which progressively becomes more and more complex, and finally, fills a subset of the plane chaotically. Starting from any initial values of the two capital supplies, the time sequence of subsequent values approaches this attractor asymptotically.

But the second two-dimensional model is our main goal in this paper. And for this model, the bifurcation diagram is shown in Figure 5.

For some values of the various parameters, we find a single basin, with a chaotic attractor. The attractor portrait for one such case is shown in Figure 6.

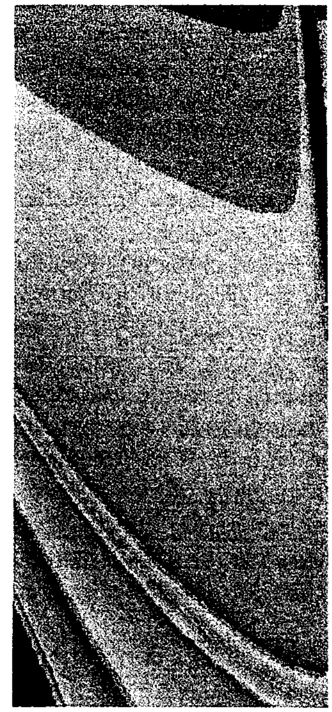

Figure 7. The two basins of attr
α_S set to 17.5 and α_N to 1.5. In
than 0.02 and 0.01 as in Figure 6.
them comprise the other basin,
attractor, in terms of number of i

Note that this attractor is
indicating that the one-dim
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For other values of the p
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This portrait has two basins
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are necessarily monostable

6. Conclusion

We introduced and develop
and studied its global dyn

ısional state space of K_N and K_S,
'he horizontal axis represents values
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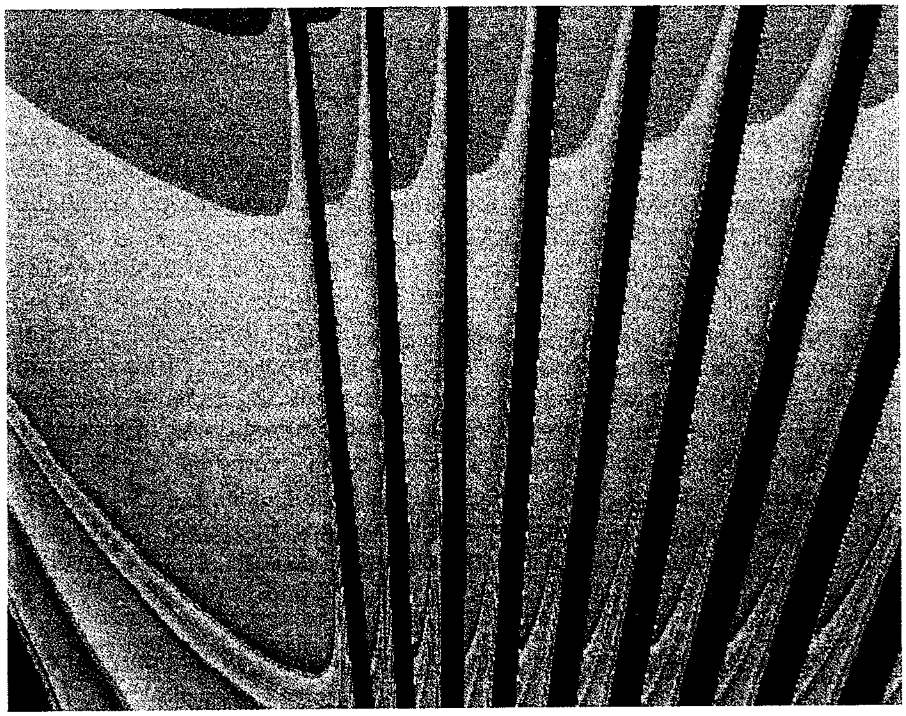

Figure 7. The two basins of attraction using the second of the two-dimensional models with α_S set to 17.5 and α_N to 1.5. In addition, the South's a_2 and c_1 are set to 0.05 and 0.04 rather than 0.02 and 0.01 as in Figure 6. The darker bands belong to one basin. The wedges between them comprise the other basin, and are shaded according to how far each point is from the attractor, in terms of number of iterations.

Note that this attractor is closely approximated by a straight line segment, indicating that the one-dimensional model is surprisingly good, at least for these values of the parameters.

For other values of the parameters, we find multistability. That is, there are two or more basins. The basin portrait for one such case is shown in Figure 7. This portrait has two basins, each containing a point attractor. The two basins are separated by a fractal boundary. This portrait is radically nonlinear, and indicates a significant difference from the one-dimensional models, which are necessarily monostable (that is, they have a single attractor).

6. Conclusion

We introduced and developed a dynamic version of the North-South model and studied its global dynamics. Our methodology was to replace the static capital endowments in the North-South model by a process of capital accumulation and depretiation through time. After showing that this leads to a well-defined dynamical system on the plane, we studied the evolution of trade and the environment through the global dynamics of the system. We showed that there is a crucial parameter which explains global dynamics: this is the regime of property rights for environmental assets in developing countries, i.e. in the region we call the South. We showed that the less well-defined are these property rights, the more chaotic is the model. We studied the particular characteristics of this chaotic system.

In a future development we hope to explore the global climate in relation with international trade. In this context, the common property resource is the planet's atmosphere, which is used as an input to production, for example, in the combustion of fossil fuels (oil). A by-product of this combustion is CO_2. In this case we would study not one but *two* separate but closely interacting dynamical systems on the plane: international trade and the biosphere (atmospheric chemistry, solar radiation, biological gas exchange, ocean dynamics, water reservoirs, climate, etc.). Especially, we will explore the greenhouse gas exchange between (1) the atmosphere, (2) human populations (which inhale oxygen and exhale carbon dioxide, both by breathing and by industrial activities), and (3) biomass and bodies of water, which act as CO_2 reservoirs.

A simple biosphere model for beginning the study of this connection is the *daisy-world model* of Watson and Lovelock. This model achieves climate regulation with two cooperating species of "daisies": black daisies (preferring cool but making warmth) and white daisies (preferring warm but making cool). One can replace one species of daisies by human industry, and by doing so extend the analysis of this paper to consider two coupled dynamical systems: the dynamical North-South system and the modified daisy-world system just described. The dynamical North-South model will be extended to three dimensions: K, L and $\bar{E}$. See [2] for this extension in a static framework.

Notes

1. See also [6].
2. See Equation (4.1.1).
3. More details are given in Section 1.3 below.
4. These are standard exogenous parameters, common to all general equilibrium models: technologies, supplies of inputs, i.e. capital and environment, and the preferences in the two regions.
5. The North-South model can be solved analytically by a single "resolving" equation [1]. This means that, knowing the exogenous parameters we can compute explicitely the equilibrium values of the model.
6. GNP is the value of the outputs minus the value of the inputs. In other words: it is the inner product of the equilibrium prices with the difference between outputs and inputs at an equilibrium.

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